Engineering Mathematics

For

Computer Science

By

THE GATE ACADEMY
A Forum of IIT / IISc Graduates

www.thegateacademy.com

✆080-40611000

Syllabus for Engineering Mathematics

Linear Algebra: Matrices, Determinants, System of Linear Equations, Eigenvalues and Eigenvectors, LU Decomposition.

Probability: Random Variables. Uniform, Normal, Exponential, Poisson and Binomial Distributions. Mean, Median, Mode and Standard Deviation. Conditional Probability and Bayes Theorem.

Calculus: Limits, Continuity and Differentiability. Maxima and Minima. Mean Value Theorem. Integration.

Previous Year GATE Papers and Analysis

GATE Papers with answer key

thegateacademy.com/gate-papers

Subject wise Weightage Analysis

thegateacademy.com/gate-syllabus

Contents

""Great thoughts speak only to the thoughtful mind, but great actions speak to all mankind."

...Emily P. Bissell

CHAPTER 1

Linear Algebra

Learning Objectives

After reading this chapter, you will know:

1. Matrix Algebra, Types of Matrices, Determinant
2. Cramer's rule, Rank of Matrix
3. Eigenvalues and Eigenvectors

Matrix

Definition

A system of "mn" numbers arranged along m rows and n columns. Conventionally, A matrix is represented with a single capital letter.

$$\text{Thus, } A = \begin{bmatrix} a_{11} & a_{12} & - & - & a_{1j} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2j} & - & - & a_{2n} \\ - & - & - & - & a_{ij} & - & - & a_{in} \\ a_{m1} & a_{m2} & - & - & - & - & - & a_{mn} \end{bmatrix}$$

$A = (a_{ij})_{m\times n}$

$a_{ij} \rightarrow i^{th}$ row, j^{th} column

Principle diagonal, Trace transpose

Types of Matrices

1. **Row and Column Matrices**
   - Row Matrix $\rightarrow [\,2\ \ 7\ \ 8\ \ 9\,] \rightarrow$ A matrix having single row is row matrix or row vector
   - Column Matrix $\rightarrow \begin{bmatrix} 5 \\ 10 \\ 13 \\ 1 \end{bmatrix} \rightarrow$ Single column (or column vector)

2. **Square Matrix**
   - Number of rows = Number of columns
   - Order of Square matrix $\rightarrow$ No. of rows or columns

   Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 6 \\ 0 & 7 & 5 \end{bmatrix}$; Order of this matrix is 3

   - **Principal Diagonal (or Main Diagonal or Leading Diagonal)**
     The diagonal of a square matrix (from the top left to the bottom right) is called as principal diagonal.

- **Trace of the Matrix**
  The sum of the diagonal elements of a square matrix.
  - tr (λ A) = λ tr(A) [λ is scalar]
  - tr (A+B) = tr (A) + tr (B)
  - tr (AB) = tr (BA)

3. **Rectangular Matrix:** Number of rows ≠ Number of columns.

4. **Diagonal Matrix:** A square matrix in which all the elements except those in leading diagonal are zero.

   Example: $\begin{bmatrix} -4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

5. **Scalar Matrix:** A Diagonal matrix in which all the leading diagonal elements are same.

   Example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

6. **Unit Matrix (or Identity Matrix):** A Diagonal matrix in which all the leading diagonal elements are '1'.

   Example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. **Null Matrix (or Zero Matrix):** A matrix is said to be Null Matrix if all the elements are zero.

   Example: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. **Symmetric and Skew Symmetric Matrices**
   - For symmetric, $a_{ij} = a_{ji}$ for all i and j. In other words $A^T = A$
     Note: Diagonal elements can be anything.
   - Skew symmetric, when $a_{ij} = -a_{ji}$ In other words $A^T = -A$
     Note: All the diagonal elements must be zero.

   Symmetric $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ Skew symmetric $\begin{bmatrix} 0 & -h & g \\ h & 0 & -f \\ -g & f & 0 \end{bmatrix}$

   Symmetric Matrix $A^T = A$ Skew Symmetric Matrix $A^T = -A$

9. **Triangular matrix**
   - A square matrix is said to be "upper triangular" if all the elements below its principal diagonal are zeros.
   - A square matrix is said to be "lower triangular" if all the elements above its principal diagonal are zeros.

   $\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$ $\begin{bmatrix} a & 0 & 0 \\ g & b & 0 \\ f & h & c \end{bmatrix}$

   Upper Triangular Matrix Lower Triangular Matrix

10. **Orthogonal Matrix:** If $A.\ A^T = I$, then matrix A is said to be Orthogonal matrix.

11. **Singular Matrix:** If $|A| = 0$, then A is called a singular matrix.

12. **Conjugate of a Matrix:** Transpose of a conjugate.

13. **Unitary matrix:** A complex matrix A is called **Unitary** if $A^{-1} = A^T$
Example: Show that the following matrix is unitary
$$A = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$
Solution: Since
$$AA^T = \frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I$$
We conclude $A^T = A^{-1}$. Therefore, A is a unitary matrix

14. **Hermitian Matrix:** It is a square matrix with complex entries which is equal to its own conjugate transpose.
$A^\theta = A$ or $a_{ij} = \bar{a}_{ij}$
For example: $\begin{bmatrix} 5 & 1-i \\ 1+i & 5 \end{bmatrix}$
Note: In Hermitian matrix, diagonal elements → Always real

15. **Skew Hermitian Matrix:** It is a square matrix with complex entries which is equal to the negative of conjugate transpose.
$A^\theta = -A$ or $a_{ij} = -\bar{a}_{ji}$
For example = $\begin{bmatrix} 5 & 1-i \\ 1+i & 5 \end{bmatrix}$
Note: In Skew-Hermitian matrix, diagonal elements → Either zero or Pure Imaginary.

16. **Idempotent Matrix:** If $A^2 = A$, then the matrix A is called idempotent matrix.

17. **Involuntary matrix:** $A^2 = I$

18. **Nilpotent Matrix :** If $A^k = 0$ (null matrix), then A is called Nilpotent matrix (where k is a +ve integer).

19. **Periodic Matrix :** If $A^{k+1} = A$ (where, k is a +ve integer), then A is called Periodic matrix.
If k =1 , then it is an idempotent matrix.

20. **Proper Matrix :** If $|A| = 1$, matrix A is called Proper Matrix.

Equality of Matrices

Two matrices can be equal if they are of
(a) Same order
(b) Each corresponding element in both the matrices are equal

Addition and Subtraction of Matrices

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \pm \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 \pm a_2 & b_1 \pm b_2 \\ c_1 \pm c_2 & d_1 \pm d_2 \end{bmatrix}$$

Rules

1. Matrices of same order can be added
2. Addition is cummulative →A+B = B+A
3. Addition is associative →(A+B) +C = A+ (B+C) = B + (C+A)

Multiplication of Matrices

Condition: Two matrices can be multiplied only when number of columns of the first matrix is equal to the number of rows of the second matrix. Multiplication of $(m \times n)$ and $(n \times p)$ matrices results in matrix of $(m \times p)$dimension $\left[\begin{matrix} m \times n \\ n \times p \end{matrix} = m \times p\right]$

Properties of multiplication

1. Let $A_{m\times n}, B_{p\times q}$ then $AB_{m\times q}$ exists $\Leftrightarrow n = p$
2. $BA_{p\times n}$exists $\Leftrightarrow q$
3. $AB \neq BA$
4. $A(BC) = (AB)C$
5. $AB = 0$ need not imply either $A = 0$ or $B = 0$

Multiplication of Matrix by a Scalar: Every element of the matrix gets multiplied by that scalar.

Determinant

An n^{th} order determinant is an expression associated with $n \times n$ square matrix.

If $A = [a_{ij}]$, Element a_{ij} with i^{th} row, j^{th} column.

For $n = 2$, $D = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$

Determinant of "Order n"

$$D = |A| = \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} \\ a_{21} & - & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn} \end{vmatrix}$$

Minors & Cofactors

- The minor of an element is the determinant obtained by deleting the row and the column which intersect that element.

  $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

  Minor of $a_1 = \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$

- Cofactor is the minor with "proper sign". The sign is given by $(-1)^{i+j}$ (where the element belongs to i^{th} row, j^{th} column).

A2 = Cofactor of $a_2 = (-1)^{1+2} \times \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$

Cofactor matrix can be formed as $\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

In General

- $a_i A_j + b_i B_j + c_i C_j = \Delta \quad \text{if } i = j$
- $a_i A_j + b_i B_j + c_i C_j = 0 \quad \text{if } i \neq j$

{a_i , b_i, c_i are the matrix elements and A_i , B_i, C_i are corresponding cofactors. }

Note: Singular matrix: If $|A| = 0$ then A is called singular matrix.

Nonsingular matrix: If $|A| \neq 0$ other A is called Non − singular matrix.

Properties of Determinants

1. A determinant remains unaltered by changing its rows into columns and columns into rows.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ i.e., } \det A = \det A^T$$

2. If two parallel lines of a determinant are inter-changed, the determinant retains it numerical values but changes in sign. (In a general manner, a row or column is referred as line).

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$$

3. Determinant vanishes if two parallel lines are identical.
4. If each element of a line be multiplied by the same factor, the whole determinant is multiplied by that factor. [Note the difference with matrix].

$$P\begin{vmatrix} a_1 & Pb_1 & c_1 \\ a_2 & Pb_2 & c_2 \\ a_3 & Pb_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

5. If each element of a line consists of the m terms, then determinant can be expressed as sum of the m determinants.

$$\begin{vmatrix} a_1 & b_1 & c_1 + d_1 - e_1 \\ a_2 & b_2 & c_2 + d_2 - e_2 \\ a_3 & b_3 & c_3 + d_3 - e_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix}$$

6. If each element of a line be added equi-multiple of the corresponding elements of one or more parallel lines, determinant is unaffected.

   Example: By the operation, $R_2 \to R_2 + pR_1 + qR_3$, determinant is unaffected.
7. Determinant of an upper triangular/ lower triangular/diagonal/scalar matrix is equal to the product of the leading diagonal elements of the matrix.
8. If A & B are square matrix of the same order, then $|AB| = |BA| = |A||B|$.
9. If A is non-singular matrix, then $|A^{-1}| = \frac{1}{|A|}$.
10. Determinant of a skew symmetric matrix (i.e., $A^T = -A$) of odd order is zero.
11. If A is a unitary matrix or orthogonal matrix (i.e., $A^T = A^{-1}$) then $|A| = \pm 1$.
12. If A is a square matrix of order n then $|kA| = k^n|A|$.
13. $|I_n| = 1$ (I_n is the identity matrix of order n).

Multiplication of Determinants

- The product of two determinants of same order is itself a determinant of that order.
- In determinants we multiply row to row (instead of row to column which is done for matrix).

Comparison of Determinants & Matrices

Although looks similar, but actually determinant and matrix is totally different thing and its technically unfair to even compare them. However just for reader's convenience, following comparative table has been prepared.

Determinant	Matrix
No. of rows and columns are always equal	No. of rows and column need not be same (square/rectangle)
Scalar Multiplication: Elements of one line (i.e., one row and column) is multiplied by the constant	**Scalar Multiplication:** All elements of matrix is multiplied by the constant
Can be reduced to one number	Can't be reduced to one number
Interchanging rows and column has no effect	Interchanging rows and columns changes the meaning all together
Multiplication of 2 determinants is done by multiplying rows of first matrix & rows of second matrix	Multiplication of the 2 matrices is done by multiplying rows of first matrix & column of second matrix

Transpose of Matrix

Matrix formed by interchanging rows & columns is called the transpose of a matrix and denoted by A^T.

Example: $A = \begin{bmatrix} 1 & 2 \\ 5 & 1 \\ 4 & 6 \end{bmatrix}$ Transpose of A= Trans (A)= $A' = A^T = \begin{bmatrix} 1 & 5 & 4 \\ 2 & 1 & 6 \end{bmatrix}$

Note:

- $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) =$ symmetric matrix + skew-symmetric matrix.
- If A & B are symmetric, then AB+BA is symmetric and AB−BA is skew symmetric.
- If A is symmetric, then A^n is symmetric (n=2, 3, 4…….).
- If A is skew-symmetric, then A^n is symmetric when n is even and skew symmetric when n is odd.

Adjoint of a Matrix

Adjoint of A is defined as the transposed matrix of the cofactors of A. In other words,
Adj (A) = Trans (cofactor matrix)

Determinant of the square matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The matrix formed by the cofactors of the elements in A is

$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} \rightarrow$ Also called as cofactor matrix

Then transpose of $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$ = Adj (A)

Inverse of a Matrix

- $A^{-1} = \frac{\text{Adj } A}{|A|}$
- $|A|$ must be non-zero (i.e. A must be non-singular).
- Inverse of a matrix, if exists, is always unique.
- If it is a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse will be $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Drill Problem: Prove $(A\,B)^{-1} = B^{-1}\,A^{-1}$

Proof: RHS = $(B^{-1}\,A^{-1})$

Pre-multiplying the RHS by AB, $(A\,B)\,(B^{-1}\,A^{-1}) = A\,(B.\,B^{-1})\,A^{-1} = I$

Similarly, Post-multiplying the RHS by AB, $(B^{-1}\,A^{-1})\,(A\,B) = B^{-1}(A^{-1}A)\,B = B^{-1}\,B = I$

Hence, AB & $B^{-1}\,A^{-1}$ are inverse to each other

Important Points

1. $IA = AI = A$, (Here A is square matrix of the same order as that of I)
2. $0\,A = A\,0 = 0$, (0 is null matrix)
3. If $AB = 0$, then it is not necessarily that A or B is null matrix.
   Also it doesn't mean $BA = 0$
   Example: $AB = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
4. If the product of two non-zero square matrix A & B is a zero matrix, then A & B are singular matrix.
5. If A is non-singular matrix and A B=0, then B is null matrix.
6. $AB \neq BA$ (in general) → Commutative property is not applicable
7. $A(BC) = (A\,B)C$ → Associative property holds.
8. $A(B+C) = AB + AC$ → Distributive property holds.
9. $AC = AD$, doesn't imply $C = D$ [Even when $A \neq 0$].
10. $(A+B)^T = A^T + B^T$
11. $(AB)^T = B^T\,A^T$
12. $(AB)^{-1} = B^{-1}\,A^{-1}$
13. $A\,A^{-1} = A^{-1}A = I$
14. $(kA)^T = k\,A^T$ (k is scalar, A is vector)
15. $(kA)^{-1} = k^{-1}\,A^{-1}$ (k is scalar, A is vector)
16. $(A^{-1})^T = (A^T)^{-1}$
17. $(\overline{A^T}) = (\bar{A})^T$ (Conjugate of a transpose of matrix = Transpose of conjugate of matrix)
18. If A non-singular matrix A is symmetric, then A^{-1} is also symmetric.
19. If A is a orthogonal matrix , then A^T and A^{-1} are also orthogonal.
20. If A is a square matrix of order n then
    (i) $|\text{adj } A| = |A|^{n-1}$
    (ii) $|\text{adj }(\text{adj } A)| = |A|^{(n-1)^2}$
    (iii) $\text{adj }(\text{adj } A) = |A|^{n-2}A$